

Stable reduction for curves.

Last time: $\mathcal{M}_{g,n}^{\text{all}}$ is an alg stack finite type over $\text{Spec } \mathbb{Z}$.

$$\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n} \subseteq \underbrace{\mathcal{M}_{g,n}^{\text{pre}} \subseteq \mathcal{M}_{g,n}^{\text{nodal}}}_{\text{moduli of prestable curves}} \subseteq \mathcal{M}_{g,n}^{\text{all}}.$$

(we don't require $\omega_X(p_1 + \dots + p_n)$ is ample)
(These inclusions are all as open substacks).

Today: Prove that $\overline{\mathcal{M}}_{g,n}$ is [proper] alg stack over $\text{Spec } \mathbb{Z}$.
Key idea: Valuative criterion for properness.

Thm (Stable reduction)

Let R be a DVR, $K = \text{Frac}(R)$. Let $(C_k, s_{1,k}, \dots, s_{n,k}) \rightarrow \text{Spec } K$ be a family of stable curves of genus g .

Then there exists a finite ext'n of DVRs $R \rightarrow R'$ and a family of stable curves $(C', s_1, \dots, s_n) \rightarrow \text{Spec } R'$ s.t.

it fits in the following diagram:

$$\begin{array}{ccc} C_k' & \xrightarrow{\quad} & \exists C_k \\ \downarrow & \swarrow & \downarrow \\ C_k & \xrightarrow{\quad} & \text{Spec } K' \\ \downarrow & \searrow & \downarrow \\ \text{Spec } K & \xrightarrow{\quad} & \text{Spec } R' \end{array}$$

This is a Cartesian square.
If you have a filling for $\text{Spec } K' \rightarrow \text{Spec } R'$ then $\dots$ for K'/K

(Let's assume $n=0$)

Proof Step 0 Reduce to the case C_k is smooth over K .

Suppose we know stable reduction for smooth curves.

Suppose C_k is a nodal curve, after a finite base change,

we can assume nodes of C_k are K -points.

Let $(\tilde{C}_k, \tilde{p}_1, \dots, \tilde{p}_m) \rightarrow C_k$ be the normalization (C_k has m nodes)

For each conn. component of $(\tilde{C}_k, \tilde{p}_1, \dots, \tilde{p}_m)$,

we can find a stable filling.

Then, gluing these stable fillings together along the sections (and possibly

this gives us a stable filling of $C_k \rightarrow \text{Spec } K$.

finite ext'n of DVRs)

(Sections are $\text{Spec } R' \xrightarrow{s_i} C_i' \rightarrow C_i' \sqcup C_i' \xrightarrow{s_i \circ s_2}$

Step 1 Now assume C_k is smooth.

Find any flat extension $C \rightarrow \text{Spec } R$.

Note that $\omega_{C_k/\text{Spec } K}^{\otimes 3}$ is very ample, so we have a closed embedding $C_k \hookrightarrow \mathbb{P}_R^{5g-6}$.

Then let C be the closure of $C_k \hookrightarrow \mathbb{P}_R^{5g-6}$.

Claim: $C \rightarrow \text{Spec } R$ is flat.

Reason: By local criterion of flatness (when the base is a local regular local ring),

then flat $\iff C$ doesn't have any embedded points mapping to

the closed pt of $\text{Spec } R$.

($\text{Tor}_1^R(k, -) = 0$).

Step 2 Modify the central fiber of $C \rightarrow \text{Spec } R$ so that

$(C_0)_{\text{red}}$ is nodal.

central fiber = $C \times_R (R/m)$.

Take a log resolution of (C, C_0) . (blow-up closed points of C_0).

Then the result would satisfy above conditions.

[(C, C_0) is a surface, so resol. of sing always exists].

Step 3 Take a finite ext'n of DVRs $R \rightarrow R'$

s.t. $C_{R'} = C \times_R R'$ is a (flat) family of curves over R'

and the central fiber is reduced and nodal

(Assume $\text{char} = 0$).

Let $p \in (C_0)_{\text{red}}$ be a node. This means that (formally) locally at p ,

we can write the map $C \rightarrow \text{Spec } R$ as

$\text{Spec } K[x, y] \rightarrow \text{Spec } R \xrightarrow{t} \text{Spec } R'$ (after taking formal nbhd).

$x^a y^b \longleftarrow t$ (uniformizer)

for some $a, b \geq 0$.

The local equation for the central fiber is $t - x^a y^b$.

Good: " $t - xy$ "

Not good notations.

Case 1: $a=0$ or $b=0$. (Assume $b=0$).

We take $R' = R$ but $R \rightarrow R'$

$t \mapsto t^a$

then the local equation of any $p' \in C_{R'}$ (p' is a preimage of p)

is a factor of $t^a - x^a = \prod_{i=0}^{a-1} (t - \zeta^i x)$

assume that ζ^i exists and are distinct ($\text{char} = 0$).

Thus, after base change, the central fiber (around p)

becomes a union of a reduced components.

Case 2 $a, b \neq 0$.

Assume $\gcd(a, b) = 1$. (If not, do Case 1 for $\gcd(a, b)$).

Let's take $R' = R[t^{1/b}]$ but $\text{Spec } R' \rightarrow \text{Spec } R$

$s^{ab} = t \longleftarrow t$

$R' \cong R[s]/(s^{ab} - t)$.

$R' = R[t^{1/b}]$.

Goal: $(\tilde{C})_{\text{red}}$ has reduced nodal central fiber.

$\tilde{C}$ normalization.

$C_{R'} \rightarrow C$

$\downarrow$

$\text{Spec } R' \rightarrow \text{Spec } R$

the (completed) local ring of $\tilde{p}$ at $(\tilde{C})_0$ looks like

the normalization of $K[[x, y, s]]/(s^{ab} - x^a y^b)$.

Key: compute normalization of $K[[x, y, t]]/(t^{ab} - x^a y^b)$

(show that this local equation factorizes).

$\gcd(a, b) = 1 \implies mb - na = 1$ ($m, n \geq 0$ int.)

then: $u := \frac{t^{mb}}{x^{mb} y^n}$ satisfies $u^a = y$.

$v := \frac{x^{na} y^n}{t^{na}}$ $\therefore v^b = x$.

$u, v \in \text{Integral closure of } K[[x, y, s]]/(t^{ab} - x^a y^b)$.

Claim: After joining u, v , the ring is integrally closed.

$K[[u, v, x, y, t]]/(u^a = y, v^b = x, uv = t)$

$\cong K[[u, v]]$ which is int. closed.

and the local ring map is $\text{Spec } K[[u, v]] \rightarrow \text{Spec } R$

$uv \longleftarrow t$

(hence the local equation of central fiber is $t - uv$) reduced!

Step 4 Make the central fiber stable.

Let $C' \rightarrow C$ be a (minimal) resolution of singularities,

then $C' \rightarrow \text{Spec } R$ is a family of nodal curves.

If C_0 is not stable, then by Prop 5.3.20, we can

contract rational tails and bridges

(-1)-curve. (-2)-curve.

Castelnuovo's theorem.

After contracting all (-1)-curves, we get "minimal model" of C' over R

(K_C/R is nef).

Then, we may take the relative canonical model of C'/R to contract (-2)-curves.

self-intersection is -1 b/c

□

Remark In char p case, this method doesn't work

$t - x^p \xrightarrow{t \mapsto s^p} s^p - x^p = (s - x)^p$

mult p

mult p .

Instead, we choose a fin. ext'n K'/K such that

C_k has a K' -point and $\text{Pic}(C_k)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$ for

suff large prime ℓ .

This will force central fiber is reduced & nodal. (Artin, Winters '71)

Degenerate fibers and stable

reduction of curves.

Examples

1. (Colliding marked points).

C sm. curve. $\mathcal{C} := C \times A^1$.

Fix $p \in C$. Let $s_1: A^1 \rightarrow \mathcal{C}$ be the const section

$t \mapsto (p, t)$

$s_2: A^1 \rightarrow \mathcal{C}$ be a section s.t.

s_1 & s_2 intersects transversally at $(p, 0)$.

$\tilde{s}_1$ $\tilde{s}_2$ $\mathcal{C}$

A^1

0

Q: What is the stable limit of this family over $0 \in A^1$?

A: Blow-up $(p, 0) \in \mathcal{C}$.

$\tilde{s}_1$ $\tilde{s}_2$ $B \mid \mathcal{C}$

$E \cong \mathbb{P}^1$

A^1

Then this is the stable limit, b/c the central fiber is now

$\mathcal{C}$

$\Rightarrow$ thus stable.

$\mathbb{P}^1$

2. (A family of nodes degenerating to a cusp).

$\mathcal{C} \rightarrow A^1$, given by $y^2 - x^3 + tx^2$.

$\mathcal{C}$ A^1

0

Q: What is the stable limit?

We can view the family $\mathcal{C}$ as follows:

It's a stable family with 2 marked sections

$(\mathcal{C}', s_1, s_2) \rightarrow A^1$.

and $\mathcal{C}$ is constructed by gluing s_1, s_2 in $\mathcal{C}'$.

s_1 s_2 $\mathcal{C}'$

A^1

0

Blow-up $\mathcal{C}'$ to get

$\tilde{s}_1$ $\tilde{s}_2$ $B \mid \mathcal{C}'$

E

Then glue $\tilde{s}_1, \tilde{s}_2$ together to get a stable family

$\mathcal{C}_0$ $\mathcal{C}'$

E

A^1

0

So the stable limit is the union of $\mathcal{C}_0$ and a nodal rat'l curve.

Rely on some gluing theory.